B.A/B.Sc.6th Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH6DSE31
(Mathematical Modelling)

Time:3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions $10\times 2 = 20$

(a)	State Poisson axioms of arrivals of a queueing system.	[2]
(b)	What are abiotic factors of an ecosystem?	[2]
(c)	What is mutualism?	[2]
(d)	What are the advantages of mathematical modelling?	[2]
(e)	Define an ecosystem. What are the different components of an ecosystem?	[2]
(f)	What do you mean by effective arrival rate of a queueing system with finite capacity?	[2]
(g)	What are the state variables for the dynamical models of ecosystem?	[2]
(h)	What is asymptotic stability of the equilibrium state of the differential equation $\frac{dx}{dt} = f(x)$?	[2]
(i)	Define stability and instability of a fixed point of the difference equation $x_{n+1} = f(x_n)$.	[2]
(j)	Find the probability of queue size $\geq N$ for queueing model $(M/M/1):(\infty/FCFS/\infty)$.	[2]
(k)	What are the drawbacks of the Malthus model for a single-species population?	[2]
(l)	Write down the confined exponential growth model for a single-species population.	[2]
(m)	Define traffic intensity of a queueing system.	[2]
(n)	Explain with an example the concept of time-delay in the mathematical modelling of growth of a population.	[2]
(o)	Write down the logistic model of population growth explaining the different terms involved in it.	[2]

2. Answer any four questions $4\times5 = 20$

(a) Investigate the asymptotic stability of the equilibrium points of the model equation $\frac{dx}{dt} = rx(1-x/k)$. [5]

(b) The growth of a population satisfies the following difference equation [5]

$$x_{n+1} = \frac{kx_n}{b+x_n},\ b,k>0.$$

Find the steady state (if any). If so, is that stable?

(c) Investigate the stability of the steady state of the logistic difference equation $x_{n+1} = rx_n(1-x_n)$. [5]

(d) A population is governed by the equation $\frac{dx}{dt} = x(e^{3-x}-1)$. Find all equilibria and determine their stability. [2+3]

(e) Discuss generalised least squares estimator. [5]

(f) Show that the non-trivial equilibrium (x^*, y^*) of the predator-prey model [5]

$$\frac{dx}{dt} = x(1-x/k) - \frac{axy}{x+A}$$
$$\frac{dy}{dt} = y\left(\frac{ax}{x+A} - \frac{aB}{A+B}\right)$$

is unstable if $k > A+2B$ and asymptotically stable if $B<k<A+2B$.

3. Answer any two questions $2\times10 = 20$

(a) Discuss Malthus model equation of population growth $\frac{dN}{dt} = rN$. Interpret the equation when the sign of r is reversed. [7+3]

(b) (i) Explain the modelling of a system in discrete time. [6]

(ii) Derive the mathematical model of traffic flows. [4]

(c) (i) Discuss generalised least squares estimator. [5]

(ii) The growth of a population satisfies the following difference equation [5]

$$x_{n+1} = \frac{kx_n}{b+x_n},\ b,k>0.$$

Find the steady state (if any). If so, is that stable?

(d) (i) Derive the steady state difference equations for the queueing model $(M/M/1):(N/FCFS/\infty)$. [5]

(ii) A population satisfies the growth equation $x_{n+2} - 2x_{n+1} + 2x_n = 0$. Find the population in n-th generation. Also, find the steady state. [5]

(B.A/B.Sc. 6th Semester (General) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH6DSE32
(Industrial Mathematics)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions $10\times2 = 20$

(a) Define and explain convolution kernel or pointspreadfunction. [2]

(b) Write a brief note on X-ray tomography. [2]

(c) What is a space of bounded measurable functions? Set an example. [2]

(d) Find the Radon transformation of the body given by [2]

$$E(x,y) = \begin{cases} 1, & 1 \le x^2 + y^2 \le 4 \\ 0, & everywhere\ else. \end{cases}$$

(e) State Beer's law. [2]

(f) Interpret Dirac delta function physically [2]

(g) State Fourier Slice theorem. [2]

(h) Why optoacoustic tomography is important to study back projections? [2]

(i) What is Shepp–Logan filter? [2]

(j) State Rayleigh–Plancherel theorem. [2]

(k) What is W-interpolation of a discrete function? [2]

(l) Why we need to include the idea of Affine space? Explain with examples. [2]

(m) Plot the Fourier transformation of $f(x) = e^{-|x|}$. [2]

(n) Define full width half maximum of the function φ. [2]

(o) Establish the formula of low pass cosine filter. [2]

2. Answer any four questions $4 \times 5 = 20$

(a) Construct the radiative transfer equation for X-rays. [5]

(b) (i) What does an attenuation coefficient measure? [2]

(ii) Why Beer's law is a plausible model for X-ray attenuation? [3]

(c) Evaluate the integral $\int_{s=-\sqrt{1-t^2}}^{\sqrt{1-t^2}} (1 - \sqrt{t^2 - s^2}) ds.$ [5]

(d) If a function g is defined by $g(x, y) = f(x - a, y - b)$ and the function h by $h(x, y) = f(cx, cy)$ where f is a function defined in a plane, a and b are arbitrary real numbers and $c > 0$ is a positive real number. Then for all real numbers t and θ prove that [5]

$$\mathcal{R}g(t, \theta) = \mathcal{R}f(t - a\cos\theta - b\sin\theta\,, \theta) \;\; and$$

$$\mathcal{R}h(t, \theta) = \left(\frac{1}{c}\right).\mathcal{R}f(ct, \theta)$$

where $\mathcal{R}$ stands for Radon transform.

(e) Apply the Rayleigh-Plancherel theorem to the function $f(x) = e^{-|x|}$ in order to evaluate the integral [5]

$$\int_{-\infty}^{\infty} \frac{1}{(1 + \omega^2)^2} d\omega.$$

(f) State and prove Nyquist's theorem. [5]

3. Answer any two questions $2 \times 10 = 20$

(a) (i) The line $\ell_{\frac{1}{2}, \frac{\pi}{6}}$ has the standard parameterization [5]

$$x = \frac{\sqrt{3}}{4} - \frac{s}{2} \text{and} \;\; y = \frac{1}{4} + \frac{\sqrt{3}}{2} s, for \; -\infty < s < \infty$$

Find the values of s at which this line intersects the unit circle.

(ii) Now define f by $f(x,y) = \begin{cases} x, & x^2+y^2 \le 1 \\ 0, & x^2+y^2 > 1 \end{cases}$. Compute $\mathcal{R}f\left(\frac{1}{2},\frac{\pi}{6}\right)$. [5]

(b) (i) Compute the full width half maximum of the Lorentzian signal [5]

$g(\omega) = \frac{T_2}{1+4\pi^2 {T_2}^2(\omega-\omega_0)^2}$,

where T_2 is a constant (one of the relaxation constants related to magnetic resonanceimaging) and the signal is centered around the angular frequency ω_0.

(ii) Find a value for B so that the Gaussian signal [5]

$$f(\omega) = T_2 e^{-B(\omega-\omega_0)^2}$$

has the same as the Lorentzian signal as in (i).

(c) Prove the identity [10]

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos M\theta = \frac{\sin(\left(M+\frac{1}{2}\right)\theta)}{2.\sin\frac{\theta}{2}}$$

for all natural numbers M and all real θ.

(d) For a given $m \times n$ matrix $A$and a given vector $\boldsymbol{p}$ in $\mathbb{R}^m$, let the function$F\colon \mathbb{R}^n \to \mathbb{R}$be defined by [10]

$F(\boldsymbol{x}) \coloneqq \|A\boldsymbol{x}-\boldsymbol{p}\|^2 = (A\boldsymbol{x}-\boldsymbol{p}).(A\boldsymbol{x}-\boldsymbol{p})$,for all $\boldsymbol{x}$in $\mathbb{R}^n$.

Show that the gradient vector of F satisfies

$$\nabla F(\boldsymbol{x}) = 2\,(A^T A\boldsymbol{x} - A^T\boldsymbol{p}),\ \text{for all } \boldsymbol{x} \text{in } \mathbb{R}^n.$$

B.A/B.Sc 6th Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH6DSE33
(Group Theory-II)

Time:3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions 10×2 = 20

(a) If $f: G \to G$ such that $f(x) = x^n$ for x in G is an automorphism, where n is some fixed integer, show that $a^{n-1} \in Z(G)$ for all a in G [2]

(b) Show that if $(Aut(G)) > 1$, then $0(G) > 2$ [2]

(c) Show that D_4 cannot be written as internal direct product of two proper subgroups. [2]

(d) If G is a finite non abelian p-group then show that $p^2|(Aut(G)|$ [2]

(e) Show that there are at most three groups of order 21 [2]

(f) Prove that an abelian group of order 2022$(n \geq 1)$ must have odd number of elements of order 2. [2]

(g) Let G be a group and A=G. Show that * defined by $a*x=axa^{-1}$,a,x in G, is a group action. [2]

(h) Let G act on G by conjugation, then show that Ker(*)=$Z(G)$ where $g*a=gag^{-1}$ for all a,g in G. [2]

(i) Let G be a group acting on a set A. If $x \in A$ andGx is the orbite of x and $y \in Gx$, then show that Gx=Gy [2]

(j) Let G be a group of order 10. Then prove that G has unique Sylow 5-subgroup. [2]

(k) Let G be a finite group acting on a set X, Then prove that $|O_x|$ divides the the order of G for every x in X. Where $O_x$denotes the orbit of x. [2]

(l) Prove that a group of order 100 which has a unique Sylow 2-subgroup is commutative. [2]

(m) Give an example of a group of order p^3, where p is prime number, which is not abelian. [2]

(n) If G is finite abelian group and a positive integer k divides (G) , then prove that G contains a subgroup of order k. [2]

(o) Consider S_4 acting on S ={1,2,3,4} defined by $\sigma.x = \sigma(x)$, for $\sigma \in S_4$ and $x \in S$. Find all orbits of the group action. [2]

2. Answer any four questions 4×5 = 20

(a) Prove that $Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$ [5]

(b) Let G be a finite group of order p^n where p is a prime and $n \geq 1$. Prove that any subgroup of G of order p^{n-1} is a normal subgroup of G. [5]

(c) Prove that any finite abelian group is isomorphic to the direct product of finite abelian groups of the prime-power order. [5]

(d) Find the of the point (x_0, y_0) the group action given below. [5]
Let $G=\mathbb{R}$ act $X=\mathbb{R}^2$ defined by $(t,(x,y))| \rightarrow (e^t x, y)$

(e) Prove that Inn(G) is normal in Aut(G) for a group G. [5]

(f) Determine the number of inner automorphism of the dihedral group D_n (n>3) [5]

3. Answer any two questions 2×10 = 20

(a) (i) Classify all subgroups of order 8 [5]

(ii) Let G be a finite group of order less than 60. If the order of G is not prime, then prove that G is not simple. [5]

(b) (i) Find all groups of order 28 [5]

(ii) Write G as a direct product of two of its proper subgroups, where G is a group of order 12. [5]

(c) (i) Find the number of distinct cyclic subgroups of order 10 in the group $Z_{50} \times Z_{25}$. [5]

(ii) Show that number of Aut(Q_8) is less than equal to 24. [5]

(d) (i) Let G be a group of order 45 with a normal P- subgroup order 9. Show that G is abelian. [5]

(ii) Prove that a group G of order 99 has a unique normal subgroup H of order 11 and H is a subset of $Z(G)$. [5]